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On acoustic refraction by duct shear layers

By P. N. SHANKAR
University of Maryland, College Park, Maryland 20742

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We consider the propagation of sound in a two-dimensional shear layer in a duct. Initially plane sound waves propagate in a subsonic flow whose steady velocity $U(y)$ is in the axial direction and is a function of the normal co-ordinate alone. The initial-value problem in time and space is solved within the framework of a perturbation scheme. Far from the wave front and for large time the refractive effects are considerable. However, the present results indicate that standing wave type patterns lead to a strong dependence on axial location in the duct. The paper also points out the weakness inherent in theories which attempt to treat the refraction problem without considering the source conditions.

1. Introduction

When sound waves propagate in a moving medium the main effects, in the absence of viscosity and thermal conduction, are convection and refraction. Convection is due simply to the tendency of the moving medium to carry the acoustic energy with it at its local velocity. In a uniformly moving medium of uniform density this is the only effect. However, if the velocity is not uniform, for example, in a shear layer, local convection velocities will be functions of position. This will cause pressure imbalances in directions normal to the flow direction resulting, further, in velocities normal to the flow. This refraction effect is due entirely to the velocity stratification in the medium.

The study of acoustic refraction is complicated for two reasons. The acoustic wavelengths normally encountered are usually too large to permit geometrical ray tracing methods. Secondly, the intimate connexion between convection and refraction prevents a simple separation of the two effects. In comparison, the study of optical refraction is conceptually much simpler.

Consider initially plane sound waves, generated at the station $x = 0$, propagating in the positive x direction in a flowing medium. The steady subsonic velocity $U(y)$ is axial and is a function of the normal co-ordinate alone. The flow is in a duct with rigid confining plates at $y = 0$ and $y = b$ (see figure 1). The question we ask is: what is the acoustic pressure distribution in the duct?

Pridmore-Brown (1958) and more recently Mungur & Gladwell (1969) have attempted to study this problem without specifying the initial pressure distribution. They assume separable solutions of the form $F(y) \exp k\{\kappa x - ct\}$ where $k = \omega/c$ is the wave-number in a stationary medium. Substitution into the governing equations and consideration of the boundary conditions at the walls leads to an eigenvalue problem for the unknown κ. The resulting solutions predict large refraction effects.

The above-mentioned analyses have a disconcerting feature. Since they do not consider the complete boundary-value problem it is not clear how their results can be applied to a given initial pressure distribution, say a plane distribution. Their eigenvalue problem is not of the Sturm–Liouville type and is, in fact, a non-linear eigenvalue problem. Unless the orthogonality and completeness of the eigenfunctions can be proved, and this appears difficult, it will not be possible to use these solutions for a given initial pressure distribution. Intimately related to the question of completeness is that of separability in the space variables. If the solution to the general boundary-value problem is in fact non-separable in the space variables, the separable solutions to the governing equations will be of little use in the refraction problem.

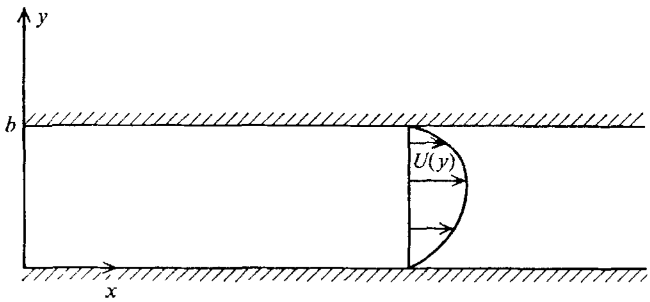

FIGURE 1. Steady velocity profile in a duct of width b.

It is the purpose of this paper to formulate a definite initial-value problem and to obtain a solution without any *a priori* assumptions regarding the form of the solution. This will be done within the framework of a perturbation scheme about a uniform flow. Under such an approximation the variables do indeed separate. Moreover, the solution exhibits features that are not apparent from the work of Pridmore-Brown and Mungur & Gladwell.

2. The initial-value problem

2.1. *Formulation*

Neglecting viscosity the linearized equations for mass and momentum conservation take the form:

$$\frac{\partial \rho}{\partial t}+U(y)\frac{\partial \rho}{\partial x}+\rho_\infty\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0, \tag{1a}$$

$$\frac{\partial u}{\partial t}+U(y)\frac{\partial u}{\partial x}+v\frac{dU}{dy} = -\frac{1}{\rho_\infty}\frac{\partial p}{\partial x}, \tag{1b}$$

$$\frac{\partial v}{\partial t}+U(y)\frac{\partial v}{\partial x} = -\frac{1}{\rho_\infty}\frac{\partial p}{\partial y}, \tag{1c}$$

where ρ_∞ is the constant ambient density and u, v, p and ρ are all first-order acoustic quantities with their usual meanings. Neglecting thermal conductivity, the energy equation and the equation of state take the form

$$\rho_\infty c_p\left(\frac{\partial T}{\partial t}+U(y)\frac{\partial T}{\partial x}\right) = \left(\frac{\partial p}{\partial t}+U(y)\frac{\partial p}{\partial x}\right), \tag{2a}$$

$$\frac{p}{p_\infty} = \frac{\rho}{\rho_\infty} + \frac{T}{T_\infty}. \tag{2b}$$

Equations $(2a)$ and $(2b)$ indicate that the pressure and density are related isentropically along particle paths and thus $(1a)$ can be rewritten

$$\frac{1}{c^2}\left(\frac{\partial p}{\partial t} + U(y)\frac{\partial p}{\partial x}\right) + \rho_\infty\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{3}$$

where $c = (\gamma R T_\infty)^{\frac{1}{2}}$ is the ambient sound speed.

Our main interest is in knowing the acoustic pressure distribution in the duct, given a harmonic plane wave sound source at $x = 0$. One could therefore assume a harmonic time dependence and attempt to solve the boundary-value problem in the space variables. Here, however, in order to avoid speculation about the application of a causality condition to the time-independent problem, we shall solve the initial-value problem in time and then look at the large time limit. The fluid is therefore assumed to be quiescent initially; the pressure at $x = 0$, the generating station, is taken to represent a harmonic, plane wave sound source whose amplitude is independent of y,

$$p(x, y, t) = 0 \quad (t \leqslant 0), \tag{4a}$$

$$p(0, y, t) = p_0 e^{-i\omega t} H(t). \tag{4b}$$

Here $H(t)$ is the Heaviside step function

$$H(t) = \begin{bmatrix} 0 & (t < 0), \\ 1 & (t > 0). \end{bmatrix} \tag{5}$$

The boundary conditions at the rigid walls are that the normal velocities must vanish there,

$$v(x, 0, t) = v(x, b, t) = 0. \tag{6}$$

The governing equations (3), $(1b)$ and $(1c)$ subject to the initial and boundary conditions $(4a)$, $(4b)$ and (6) yield the acoustic field for $t > 0$. We shall restrict our attention to the right half plane.

It is to be noted that a single equation can be written down for the pressure

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)^3 - \left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2\frac{dU}{dy}\frac{\partial^2}{\partial y\,\partial x}\right]p = 0. \tag{7}$$

2.2. *A perturbation scheme*

The governing equation (7) for the acoustic pressure is a third-order non-constant-coefficient partial differential equation. As was pointed out earlier the straightforward separation of variables approach does not appear to be useful in solving the initial- and boundary-value problem. As an alternative we shall develop a perturbation scheme about uniform flow. While such a procedure has obvious limitations, the advantage is that a solution will be obtained for a given source pressure distribution.

Let the steady axial velocity $U(y)$ be given as

$$U(y) = U_0 + \epsilon U^{(1)}(y), \tag{8}$$

where U_0 is a constant and ϵ is a small parameter. It is natural then to seek a solution in the form

$$p(x,y,t) = p^{(0)} + \epsilon p^{(1)} + \dots, \tag{9a}$$

$$u(x,y,t) = u^{(0)} + \epsilon u^{(1)} + \dots, \tag{9b}$$

$$v(x,y,t) = v^{(0)} + \epsilon v^{(1)} + \dots. \tag{9c}$$

The zeroth-order pressure $p^{(0)}$ then has to satisfy the convected wave equation

$$(1-M_0^2)\frac{\partial^2 p^{(0)}}{\partial x^2} + \frac{\partial^2 p^{(0)}}{\partial y^2} - \frac{2M_0}{c}\frac{\partial^2 p^{(0)}}{\partial x\,\partial t} - \frac{1}{c^2}\frac{\partial^2 p^{(0)}}{\partial t^2} = 0, \tag{10}$$

where $M(y) = M_0 + \epsilon M^{(1)}(y)$ is the Mach number of the steady flow. The solution satisfying (10) and the initial and boundary conditions (4), (6) is simply given by

$$p^{(0)}(x,y,t) = p_0 H\left(t - \frac{x}{c(1+M_0)}\right)\exp\left(-i\omega\left(t - \frac{x}{c(1+M_0)}\right)\right), \tag{11a}$$

$$u^{(0)}(x,y,t) = p^{(0)}/\rho_\infty c, \tag{11b}$$

$$v^{(0)}(x,y,t) = 0. \tag{11c}$$

Thus the zeroth-order solution is just a plane sound wave convected in the positive x direction. The first-order equation for the acoustic pressure then takes the form

$$(1-M_0^2)\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial y^2} - \frac{2M_0}{c}\frac{\partial^2 p^{(1)}}{\partial x\,\partial t} - \frac{1}{c^2}\frac{\partial^2 p^{(1)}}{\partial t^2} = \frac{2M^{(1)}(y)}{c}\left(\frac{\partial}{\partial t} + M_0 c\frac{\partial}{\partial x}\right)\frac{\partial p^{(0)}}{\partial x}. \tag{12}$$

Clearly the refraction effect, due to the Mach number stratification, is contained here. The initial and boundary conditions to be satisfied by the pressure perturbation are:

$$p^{(1)}(x,y,t) = 0 \quad (t < 0), \tag{13a}$$

$$p^{(1)}(0,y,t) = 0 \quad (t \geqslant 0), \tag{13b}$$

$$\frac{\partial p^{(1)}}{\partial y}(x,0,t) = \frac{\partial p^{(1)}}{\partial y}(x,b,t) = 0. \tag{13c}$$

2.3. *The transient solution*

Define the finite cosine transform and its inverse.

$$\tilde{p}^{(1)}(x,n,t) = \int_0^b p^{(1)}(x,y,t)\cos(n\pi y/b)\,dy, \tag{14a}$$

$$p^{(1)}(x,y,t) = (1/b)\,\tilde{p}^{(1)}(x,0,t) + (2/b)\sum_{n=1}^{\infty}\tilde{p}^{(1)}(x,n,t)\cos(n\pi y/b). \tag{14b}$$

Since $\partial p^{(1)}/\partial y = 0$ at $y = 0, b$, then

$$\int_0^b \frac{\partial^2 p^{(1)}}{\partial y^2}\cos\frac{n\pi y}{b}\,dy = -\frac{n^2\pi^2}{b^2}\tilde{p}^{(1)}(x,n,t).$$

Applying the cosine transform to the first-order equation (12) one obtains

$$(1-M_0^2)\frac{\partial^2 \tilde{p}^{(1)}}{\partial x^2} - \frac{2M_0}{c}\frac{\partial^2 \tilde{p}^{(1)}}{\partial x\,\partial t} - \frac{1}{c^2}\frac{\partial^2 \tilde{p}^{(1)}}{\partial t^2} - \frac{n^2\pi^2}{b^2}\tilde{p}^{(1)} = \frac{2a_n}{c}\left(\frac{\partial}{\partial t} + cM_0\frac{\partial}{\partial x}\right)\frac{\partial p^{(0)}}{\partial x}, \tag{15}$$

where a_n is the Fourier coefficient of the Mach number perturbation,

$$a_n = \int_0^b M^{(1)}(y)\cos(n\pi y/b)\,dy. \tag{16}$$

Since we are considering the initial-value problem in time it is convenient to introduce the Laplace transform and its inverse defined by

$$\hat{\tilde{p}}^{(1)}(x,n,\sigma) = \int_0^\infty \tilde{p}^{(1)}(x,n,t)\,e^{-\sigma t}\,dt, \tag{17a}$$

$$\tilde{p}^{(1)}(x,n,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \hat{\tilde{p}}^{(1)}(x,n,\sigma)\,e^{\sigma t}\,d\sigma. \tag{17b}$$

Applying the transform to (15) an ordinary differential equation is obtained for $\hat{\tilde{p}}^{(1)}$.

$$(1-M_0^2)\frac{d^2\hat{\tilde{p}}^{(1)}}{dx^2} - \frac{2M_0\sigma}{c}\frac{d\hat{\tilde{p}}^{(1)}}{dx} - \left(\frac{\sigma^2}{c^2}+\frac{n^2\pi^2}{b^2}\right)\hat{\tilde{p}}^{(1)} = -\frac{2a_n}{c}\frac{p_0}{c(1+M_0)^2}\frac{\sigma^2}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{c(1+M_0)}\right\}. \tag{18}$$

The solution for $n=0$ and $n \neq 0$ have to be considered in turn.

$n=0$. The straightforward solution of (18) subject to the conditions at $x=0$ yields

$$\hat{\tilde{p}}^{(1)}(x,0,\sigma) = a_0\frac{p_0 x}{c(1+M_0)^2}\frac{\sigma}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{c(1+M_0)}\right\}, \tag{19}$$

which appears to grow with x. Now, one can go through the formal procedure for treating this singular perturbation. However, in this case, this is unnecessary as the non-uniformity can be corrected by inspection. The $n=0$ 'mode' corresponds to simple convection as this is just the effect of an additional uniform velocity $\epsilon(a_0c/b)$ to the uniform flow cM_0. Thus it would appear that one need only correct the zeroth-order solution by this additional uniform velocity in order to account for this perturbation. That this is so is confirmed by expanding the Laplace transform of the zeroth-order solution with this term added to the uniform Mach number.

$$\frac{p}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{c(1+M_0+(\epsilon a_0/b))}\right\} = \frac{p_0}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{c(1+M_0)}\left[1-\frac{\epsilon a_0}{b(1+M_0)}+\dots\right]\right\}$$
$$= \frac{p_0}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{c(1+M_0)}\right\}\left[1+\epsilon\frac{\sigma x a_0}{bc(1+M_0)^2}+\dots\right]. \tag{20}$$

The second term is identical to (19). Thus one can treat the problem as of convection by the mean uniform Mach number $(M_0+\epsilon(a_0/b))$ and refraction by the shear profile about this mean.

$n \neq 0$. For $n \neq 0$ the bounded solution to (18) vanishing at $x=0$ is

$$\hat{\tilde{p}}^{(1)}(x,n,\sigma) = \frac{b^2}{n^2\pi^2}\frac{2a_n}{c}\frac{p_0}{c(1+M_0)^2}\frac{\sigma^2}{\sigma+i\omega}\left[\exp\left\{-\frac{\sigma x}{c(1+M_0)}\right\} - \exp\left\{\frac{x}{1-M_0^2}\left(\frac{\sigma M_0}{c}-\left[\frac{\sigma^2}{c^2}+\frac{n^2\pi^2}{b^2}(1-M_0^2)\right]^{\frac{1}{2}}\right)\right\}\right]. \tag{21}$$

Thus to order ϵ the Laplace transform in time of the pressure is given by

$$\hat{p}(x,y,\sigma) = \frac{p_0}{\sigma+i\omega}\exp\left\{-\frac{\sigma x}{1+M_0}\left(1-\frac{\epsilon a_0}{b(1+M_0)}\right)\right\}$$
$$+\epsilon.\frac{2}{b}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi y}{b}\right)\frac{b^2}{n^2\pi^2}\frac{2a_n}{c}\frac{p_0}{c(1+M_0)^2}\frac{\sigma^2}{\sigma+i\omega}$$
$$\times\left[\exp\left\{-\frac{\sigma x}{c(1+M_0)}\right\}-\exp\left\{\frac{x}{1-M_0^2}\left(\frac{\sigma M_0}{c}-\left[\frac{\sigma^2}{c^2}+\frac{n^2\pi^2}{b^2}(1-M_0^2)\right]^{\frac{1}{2}}\right)\right\}\right]. \quad (22)$$

Using the shift and product theorems (Carslaw & Jaeger 1963) the inversion of (22) is readily accomplished, formally giving, to order ϵ,

$$p(x,y,t) = p_0 H\left\{t-\frac{x}{c(1+M_0)}\left(1-\frac{\epsilon a_0}{b(1+M_0)}\right)\right\}$$
$$\times\exp\left\{-i\omega\left(t-\frac{x}{c(1+M_0)}\left(1-\frac{\epsilon a_0}{b(1+M_0)}\right)\right)\right\}+\epsilon p_0\sum_{n=1}^{\infty}\frac{4a_n}{bn^2\pi^2(1+M_0)^2}\cos\left(\frac{n\pi y}{b}\right).\frac{b^2}{c^2}$$
$$\times\left[\int_0^t g((\tau+a),\alpha,\beta)\{\delta'(t-\tau)-i\omega\delta(t-\tau)-\omega^2\exp\{-i\omega(t-\tau)\}\}d\tau\right], \quad (23)$$

where
$$g(t,\alpha,\beta) = \frac{\alpha\beta J_1[\alpha[t^2-\beta^2]^{\frac{1}{2}}]}{[t^2-\beta^2]^{\frac{1}{2}}}H(t-\beta), \quad (24a)$$
$$a = M_0x/c(1-M_0^2), \quad (24b)$$
$$\beta = x/c(1-M_0^2), \quad (24c)$$
$$\alpha^2 = (n^2\pi^2/b^2)\,c^2(1-M_0^2). \quad (24d)$$

and J_1 is the first-order Bessel function of the first kind and $\delta(t)$ and $\delta'(t)$ are the delta function and its derivative.

2.4. *Large time solution*

Our main interest lies in the pressure distribution in the duct, far from the wave front, once the transient conditions have disappeared. The large time solution can be obtained from the transient solution (23) or directly from the transform (22). Following the latter course it is clear that the large time behaviour is determined by the only pole, $\sigma = -i\omega$. Using the theorem for large time (Carslaw & Jaeger 1963), we can write down the solution for large time, behind the wave front.

$$p(x,y,t) = p_0\exp\left\{-i\omega\left(t-\frac{x}{c(1+M_0)}\left(1-\frac{\epsilon a_0}{b(1+M_0)}\right)\right)\right\}$$
$$+\epsilon p_0\sum_{n=1}^{\infty}\frac{4}{\pi^2(1+M_0)^2}\frac{a_n}{bn^2}\frac{b^2\omega^2}{c^2}\cos\left(\frac{n\pi y}{b}\right)e^{-i\omega t}\left[\exp\left\{\frac{x}{1-M_0^2}\left(-\frac{i\omega M_0}{c}\right.\right.\right.$$
$$\left.\left.\left.+i\left[\frac{\omega^2}{c^2}-\frac{n^2\pi^2}{b^2}(1-M_0^2)\right]^{\frac{1}{2}}\right)\right\}-\exp\{i\omega x/c(1+M_0)\}\right]+O(\epsilon^2). \quad (25)$$

It is easily verified that this solution satisfies the governing equations and the boundary conditions to order ϵ.

If the generating pressure distribution is taken to be $p_0 \sin \omega t$ the mean-square pressure to order ϵ is given by

$$\frac{p^2}{p_0^2} = \tfrac{1}{2} + \epsilon \sum_{n=1}^{n^*-1} B_n \cos \mu_n + \epsilon \sum_{n=n^*}^{\infty} C_n \cos \kappa + \epsilon \sum_{n=1}^{\infty} D_n, \tag{26}$$

where n^* is the smallest integer such that $n^* > (1/\pi[1-M_0^2]^{\frac{1}{2}})(\omega b/c)$ and

$$B_n = \frac{4}{\pi^2(1+M_0)^2} \frac{a_n}{bn^2} \left(\frac{b\omega}{c}\right)^2 \cos \frac{n\pi y}{b}, \tag{27a}$$

$$\mu_n = \frac{\omega x}{c(1-M_0^2)} \left\{1 - \left[1 - \frac{n^2\pi^2}{b^2} \frac{c^2}{\omega^2}(1-M_0^2)\right]^{\frac{1}{2}}\right\}, \tag{27b}$$

$$C_n = B_n \exp\left\{-\frac{\omega x}{c(1-M_0^2)} \left[\frac{n^2\pi^2}{b^2} \frac{c^2}{\omega^2}(1-M_0^2) - 1\right]^{\frac{1}{2}}\right\}, \tag{27c}$$

$$\kappa = \omega x / c(1-M_0^2), \tag{27d}$$

$$D_n = -B_n. \tag{27e}$$

3. Discussion

The expression (25) is the solution that we were seeking. For harmonic plane wave generation at $x = 0$, the pressure distribution in the duct is given by (25). The effect of the velocity perturbation $\epsilon U^{(1)}(y)$ is two fold. First of all the convection speed is changed to the mean value of the total velocity $U_0 + \epsilon U^{(1)}(y)$. The refraction or channelling then takes place as due to the shear about this mean velocity.

The solution has a number of interesting features. It is composed of the superposition of wave modes, in each of which the variables do, indeed, separate. The same result could have been obtained by rejecting the backward propagating waves in the spatial problem obtained by assuming an $\exp(-i\omega t)$ time dependence. That the latter course is valid, is not obvious in a non-homogeneous problem. The solution also exhibits a cut-off phenomenon. For a given exciting frequency ω, the modes for which $n < (1/\pi[1-M_0^2]^{\frac{1}{2}})(b\omega/c)$, travel unattenuated. For n greater than this critical value the modes are rapidly attenuated with distance. The whole phenomenon is clearly strongly dependent on the ratio of the wavelength to the duct width.

In order to show more clearly the effects of the various parameters, the results of some typical calculations are presented in figures 2–6. The generating pressure is taken to be $p_0 \sin \omega t$ and the unperturbed Mach number M_0 is taken to be zero. Thirty terms of the series were calculated; however, 10 terms would have given sufficient accuracy to make the curves indistinguishable, on the scale presented. Figures 2(*a*) and (*b*) show the effect of the velocity profile. The linear profile with the Mach number increasing upwards tends to channel the sound downwards, as expected. With the symmetric quadratic profile, the sound is channelled symmetrically about the middle of the duct. It is to be noted that the pressure build up at $(x/b) = 10$ is much greater than at the subsequent stations shown.

The effects of increasing frequency and Mach number for a given profile, the $\frac{1}{7}$ power profile, are shown in figures 3 and 4. In figure 3, the flow is counter to

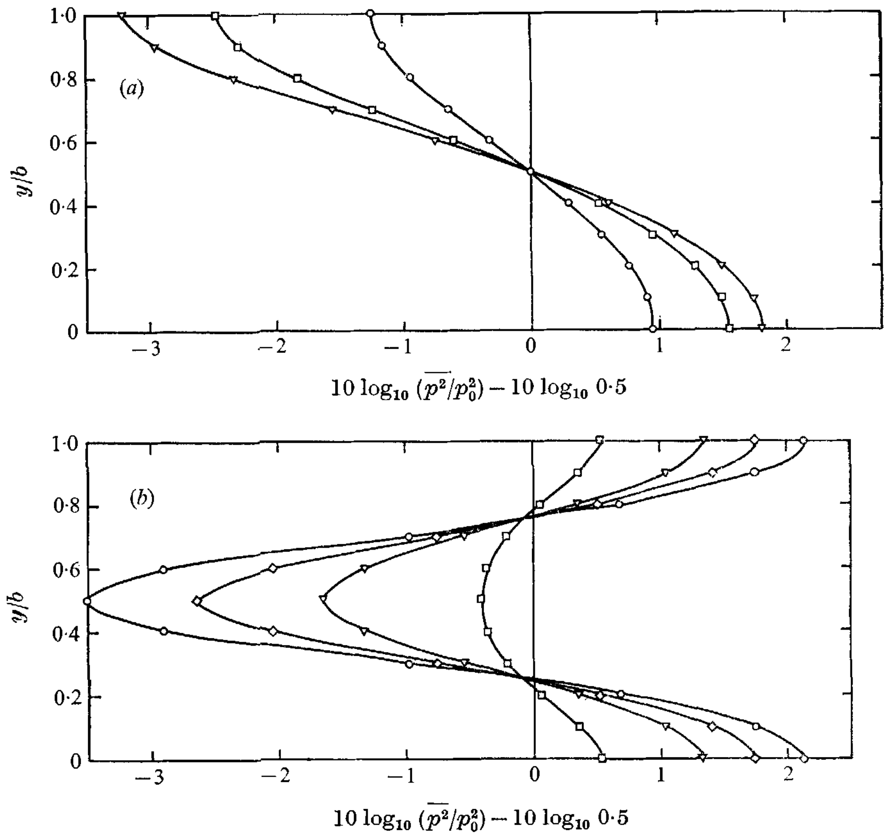

FIGURE 2. (*a*) Acoustic refraction by a linear profile. $M_0 = 0, M^{(1)}(y) = y/b, \epsilon = 0{\cdot}05, (\omega b/c) = 7$: $\bigcirc$, $x/b = 10$; $\square$, $x/b = 40$; ∇, $x/b = 70$. (*b*) Acoustic refraction by a quadratic profile. $M_0 = 0$, $M^{(1)}(y) = (y/b)\{1-(y/b)\}$, $\epsilon = 0{\cdot}3$, $(\omega b/c) = 10$: $\bigcirc$, $x/b = 10$; $\square$, $x/b = 40$; ∇, $x/b = 70$; $\diamond$, $x/b = 100$.

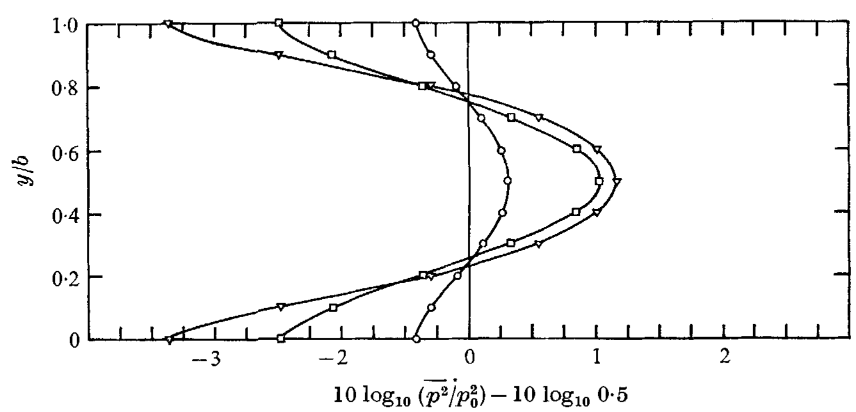

FIGURE 3. The effect of increasing frequency. $M_0 = 0$,

$$M^{(1)}(y) = \begin{cases} (2y/b)^{\frac{1}{7}} & 0 \leqslant y/b \leqslant 0{\cdot}5, \\ (2-2y/b)^{\frac{1}{7}} & 0{\cdot}5 \leqslant y/b \leqslant 1, \end{cases}$$

$\epsilon = -0{\cdot}05$, $x/b = 70$: $\bigcirc$, $\omega b/c = 10$; $\square$, $\omega b/c = 15$; ∇, $\omega b/c = 20$.

the direction of sound propagation and so the channelling is towards the centre of the duct. The refractive effect is accentuated, as expected, by increasing frequency and Mach number. However, the effect of increasing frequency decreases after a certain stage as the amplitudes of the higher travelling wave modes are attenuated by the (a_n/n^2) factor.

The details of the complicated near field are presented in figure 5. All the locations shown are within a duct width of the source. The pressure build up is not uniform in either x or y.

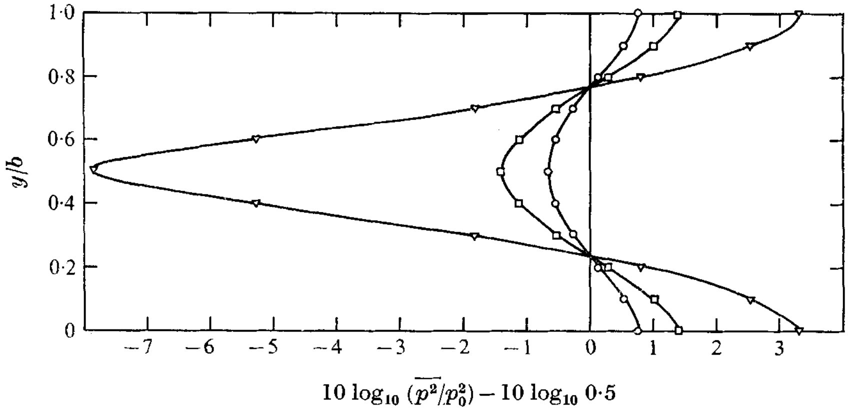

FIGURE 4. The effect of increasing Mach number for a given profile. The profile is the same as in figure 3. $\omega b/c = 15$, $x/b = 8$: ○, $\epsilon = 0{\cdot}05$; □, $\epsilon = 0{\cdot}1$; ▽, $\epsilon = 0{\cdot}3$.

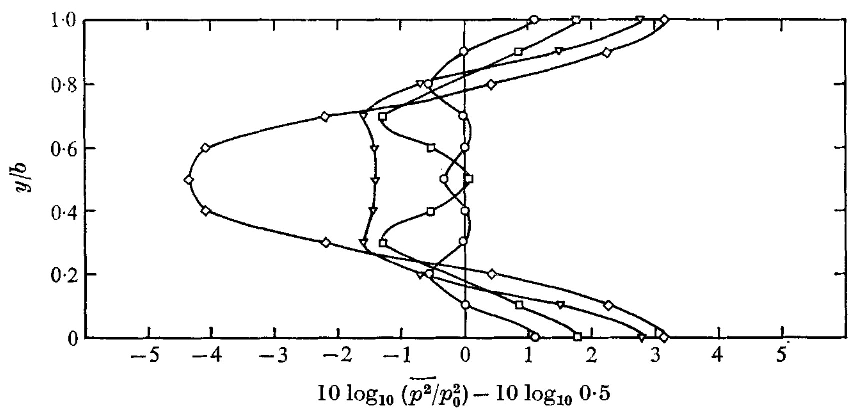

FIGURE 5. Details of the near field. Same profile as in figure 3. $\epsilon = 0{\cdot}3$, $\omega b/c = 20$: ○, $x/b = 0{\cdot}25$; □, $x/b = 0{\cdot}5$; ▽, $x/b = 0{\cdot}75$; ◇, $x/b = 1{\cdot}0$.

Inspection of the result (26) for the mean-square pressure indicates the strong dependence on the axial location. It is clear that the different wave modes can combine to yield standing wave type patterns. This effect is clearly illustrated in figure 6, where, for the given parameters, the refraction effect at $x/b = 100$ is considerably less than at $x/b = 40$ and $x/b = 70$. While one does have to be cautious, as the problem treated is that of hard wall ducts, the result would seem to indicate, qualitatively, that acoustic liners may not be equally effective at all axial locations. However, one would have to consider the liner problem separately in order to arrive at any quantitative estimates.

It would be in order to compare the results of the present approach with those of Pridmore-Brown and Mungur & Gladwell. Unfortunately, this is not possible for the reason mentioned earlier: until and unless the orthogonality and completeness of Pridmore-Brown's eigenfunctions can be proved, there appears no way of combining them to give a uniform initial pressure distribution.

Finally, we consider the likely range of validity of the present results. One cannot, *a priori*, define a precise range of validity of the perturbation scheme. One can only calculate the refractive field and regard it with caution when it appears to be of the same magnitude as the plane wave field (i.e. the calculation fails when it becomes interesting!). Thus for long wavelengths, where the refractive effects are smaller, the perturbation scheme would be applicable for higher Mach numbers than it would be for shorter wavelengths. In summary,

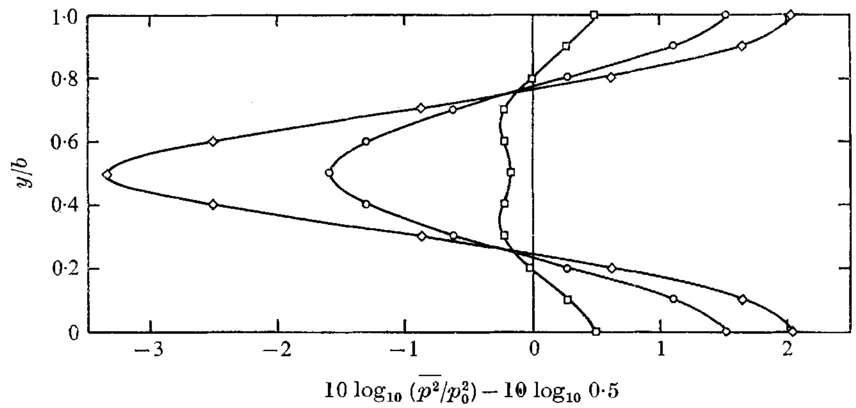

FIGURE 6. The effect of standing wave type patterns. The pressure build up appears to be decreasing with distance from the source, over the range considered. The profile is the same as in figure 3. $\epsilon = 0{\cdot}05$, $\omega b/c = 20$: $\diamond$, $x/b = 40$; $\circ$, $x/b = 70$; $\square$, $x/b = 100$.

the perturbation scheme should be regarded as a perturbation about plane wave propagation, rather than as perturbation about uniform flow. The validity of the results may be decided by the relative magnitudes of the plane wave field and the first-order refractive field. It is for this reason that the calculations presented in the figures were done by setting $M_0 = 0$ and allowing the perturbed velocity field to represent the whole velocity field.

4. Concluding remarks

An objective of this paper was to establish the importance of relating the propagation-refraction problem to the nature of the sound source. The chief results obtained here, the cut-off effect, the dependence of the mean pressure on the axial distance from the source, the standing wave type patterns, etc., were not indicated by the earlier approaches which did not consider the initial pressure distribution.

We still have to remain tentative about the question of separability in the space variables. Within the present perturbation approximation it was shown that the variables do indeed separate. However, it is obvious that the cosine transform would be inapplicable for the complete problem. It would be of interest in this connexion to investigate the properties of the eigenfunctions of Pridmore-Brown's eigenvalue problem. In determining whether they form a complete and orthogonal set or not, the question of separability will also be decided.

In conclusion it should be pointed out that the present solution has the natural limitations of a perturbation treatment. When the refraction effects become large the validity of the small perturbation approximation comes into question. One could in principle carry out the approximation systematically to higher order but we shall not consider this here.

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